LECTURES 1 AND 2

MATH MODELING II (MATH 769), TAUGHT BY PROFESSOR GREG FOREST

Notes transcribed by Gregory Herschlag

1. Notes for First Lecture (January 15)

1.1. **Simple Experiments.** To categorize visco-elastic materials we measure relationships between stress and strain. Stress will be defined as (Force)/(Area) and will be denoted as either σ or τ. Strain will be defined as (deformed length)/(undeformed length) and will be denoted as either γ or ϵ. Note that stress has units of force per area and that strain is unitless.

Two simple experiments that we will rely on are stress relaxation and creep. In stress relaxation, a strain is imposed to a material which results in a measurable stress. For example when bread dough is pressed and released, it will move back toward its original shape; this demonstrates that the bread stores stress after a deformation. In creep, a stress is imposed to a material which results in a measurable strain. For example when we pull on a spring with a certain force it will extend by a proportional amount.

We start with simple inputs in each experiment. In stress relaxation we impose strain as either a Heaviside or top-hat function of time; in creep we impose a stress in the same way. With these inputs we attempt to predict the output strain in stress relaxation, and stress in creep, as functions of time.

1.2. **Simple Models.** The simplest models we have for solids and fluids are springs and dashpot respectively (a dashpot will model a fluid element with constant viscosity). We claim that a a simple spring has the property that stress scales linearly with strain - i.e. $\sigma = G\gamma$ (note then that G has units of stress). We claim that a simple dashpot has the property that stress scales linearly with the rate of change in strain - i.e. $\sigma = \eta\dot{\gamma}$ (note that η has units of stress time). Note what happens to each element during stress relaxation in a simple top-hat experiment

$$\gamma(t) = \gamma_0[H(t-t_0) - H(t-t_1)] \tag{1.1}$$

$$\dot{\gamma}(t) = \bar{\gamma_0}[\delta(t-t_0) - \delta(t-t_1)] \tag{1.2}$$

where the bar is introduced to remind us that even though the magnitude of γ_0 has stayed the same, its units are now different. Note that for a spring, stress is the input strain scales linearly ($\sigma = G\gamma(t) = \gamma_0[H(t-t_0) - H(t-t_1)]$), but that for a dashpot we would not be able to measure stress since it only appears instantaneously. Thus the spring lends itself to stress relaxation but the dashpot does not. What will happen in creep?

Date: January 15 and 20, 2008.

Alone, the spring is purely elastic (solid like) and the dashpot is purely viscous (fluid like). In order to model visco-elastic materials we begin by creating a hierarchy of networks of these simple elements which will display properties of both solids and liquids. We do this by linking elements in parallel and in series.

1.2.1. *Two Element Models.* We begin networking our simple elements by investigating what happens when one spring and one dashpot are put in parallel and in series; these elements are known as a Voigt element and a Maxwell element respectively. The Voigt element will be considered a viscous solid because it will look like a solid in relaxation experiments, while the Maxwell element will be considered an elastic liquid because it will look like a liquid in creep.

For a Voigt element we derive a constitutive law relating stress to strain by noting the following relationships

$$\gamma = \gamma_e = \gamma_v$$
$$\sigma = \sigma_e + \sigma_v$$

where γ_e and σ_e represent the strain and stress of the spring (e for elastic), and γ_v and σ_v represent the strain and stress of the dashpot (v for viscous). γ and σ represent the strain and stress of the system. In words this says that in parallel, the dashpot and spring will be deformed by the same length and the stress that they store adds.

From this we derive a relationship as follows

$$\sigma_e = G\gamma_e,\ \sigma_v = \eta\dot{\gamma_v}$$
$$\sigma = \sigma_e + \sigma_v = G\gamma_e + \eta\dot{\gamma_v}$$
$$\sigma = G\gamma + \eta\dot{\gamma}$$

For a Maxwell element we derive a constitutive law relating stress to strain by noting the following relationships

$$\gamma = \gamma_e + \gamma_v$$
$$\sigma = \sigma_e = \sigma_v$$

In words this says that in series, the deformation of the dashpot and spring will add and the stress each undergoes will be the same.

From this we derive a relationship as follows

$$\dot{\gamma_e} = \dot{\sigma_e}/G,\ \dot{\gamma_v} = \sigma_v/\eta$$
$$\dot{\gamma} = \dot{\gamma_e} + \dot{\gamma_v} = \dot{\sigma_e}/G + \sigma_v/\eta$$
$$\eta\dot{\gamma} = \lambda\dot{\sigma} + \sigma$$

Here $\lambda = \eta/G$ and becomes a time scale for the problem (since $[\eta]/[G] = [Stress][Time]/[Stress] = [Time]$).

Lets see what happens to a Voigt element in relaxation. Appealing to equations (1.1) and (1.2) notice that the measured stress will then read $\sigma = G\gamma_0[H(t-t_0) - H(t-t_1)] + \eta\tilde{\gamma_0}[\delta(t-t_0) - \delta(t-t_1)]$. Thus in an experiment the function will look like a solid.

1.3. **Review of the Laplace Transform.** Some needed facts about the Laplace transform

$\int_a^b e^{-st} f(t)dt \equiv F(s) = \tilde{f}(s) = \mathcal{L}\{f(t)\}(s) = \mathcal{L}\{f(t)\}$

$\mathcal{L}\{\delta(t-t_0)\} = e^{-t_0 s}$

$\mathcal{L}\{H(t-t_0)\} = e^{-t_0 s}/s$

$\mathcal{L}^{-1}\{\frac{1}{s+\lambda^{-1}} e^{-t_0 s}\} = e^{-(t-t_0)s/\lambda} H(t-t_0)$

$\mathcal{L}^{-1}\{\frac{1}{s^2} e^{-t_0 s}\} = (t-t_0)H(t-t_0)$

$\mathcal{L}\{\dot{f}(t)\} = sF(s) - f(0)$
$\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)H(t-a)$

Homework: Solve for the Maxwell element in relaxation. Let the input strain be a simple top hat function. Let $\sigma(0) = \gamma(0) = 0$ and solve for stress using the Laplace transform. Create a script in some language (i.e. matlab) which takes in the time on, time off, and the magnitude of the input top-hat strain, sets η and G as parameters, and plots the stress as a function of time.

2. Notes for Second Lecture (January 20)

2.1. **Simple Models (continued).** In the Voigt and Maxwell models we had the λ parameter. In Voigt this is known as the retardation time. In Maxwell this is known as the relaxation time.

Homework: Repeat the last homework in the previous lecture for a Voigt element, this time having stress as the input.

2.1.1. *Standard Three Element Models.* The standard three element Maxwell network (Std3M) is defined as a two element Maxwell unit in parallel with a spring. The standard three element Voigt network (Std3V) is defined as a two element Voigt unit in series with a spring.

For a Std3M we can again derive a set of relationships to be used to come up with a constitutive law relating the stress and strain of the entire network

$$\lambda_1\dot{\sigma_1} + \sigma_1 = \eta_1\dot{\gamma_1},\ \lambda_1 = \eta_1/G_1$$
$$\sigma_2 = G_2\gamma_2,\ \gamma = \gamma_1 = \gamma_2,\ \sigma = \sigma_1 + \sigma_2$$

where σ_1, γ_1, η_1, and G_1 are the stress, strain, dashpot coefficient, and spring coefficient for the two element Maxwell unit, and σ_2, γ_2, and G_2 are the stress, strain, and spring coefficient of the spring in parallel. σ and γ are the stress and strain of the complete three element network.

Solving the system for σ and γ gives

$$\begin{aligned}\lambda_1\dot{\sigma} &= \lambda_1\dot{\sigma_1} + \lambda_1\dot{\sigma_2}\\ &= \eta_1\dot{\gamma_1} - \sigma_1 + \lambda_1 G_2\dot{\gamma_2}\\ &= \eta_1\dot{\gamma} - \sigma_1 + \lambda_1 G_2\dot{\gamma}\\ &= \eta_1\dot{\gamma} + \lambda_1 G_2\dot{\gamma} - (\sigma - G_2\gamma_2)\end{aligned}$$

From this we conclude

$$\lambda_1\dot{\sigma} + \sigma = \eta_1(1 + \frac{G_2}{G_1})\dot{\gamma} + G_2\gamma \tag{2.1}$$

Out of this we can find a new retardation time scale of $\lambda_2 = \lambda_1(1 + \frac{G_1}{G_2})$.

Homework: Derive a similar relationship for the Std3V. Find a relaxation time scale of $\eta_1/(G_1 + G_2)$ and a retardation time scale of $\lambda = \eta_1/G_1$.

These two networks are called "standard" because the system closes on first derivatives. Both yield interesting results in creep and relaxation. Up to differences in time scales the two models are equivalent.

We can solve the Std3M by Laplace transform for both experiments

$$\underline{\text{Relaxation:}}\tilde{\sigma}(s) = \frac{(G_1 + G_2)s + (G_2/\lambda_1)}{s + \lambda_1^{-1}}\tilde{\gamma}(s) \tag{2.2}$$

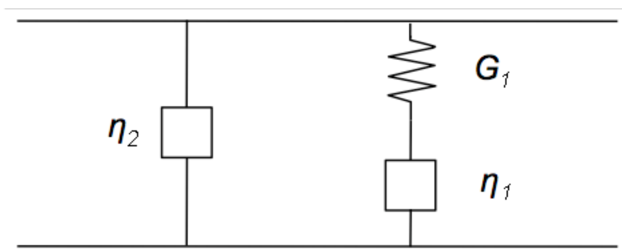
$$\underline{\text{Creep:}}\tilde{\gamma}(s) = \frac{s + \lambda_1^{-1}}{(G_1 + G_2)s + (G_2/\lambda_1)}\tilde{\sigma}(s) \tag{2.3}$$

Homework: Invert the transform and solve for the unknown in both relaxation and creep. Let the input be a simple top-hat function and assume $\sigma(0) = \gamma(0) = 0$. Plot these functions by writing a script that takes in the top-hat information and the parameter values.

Lecture Notes Math 769
January 22, 2009

Non-Standard 3-Parameter Maxwell Model

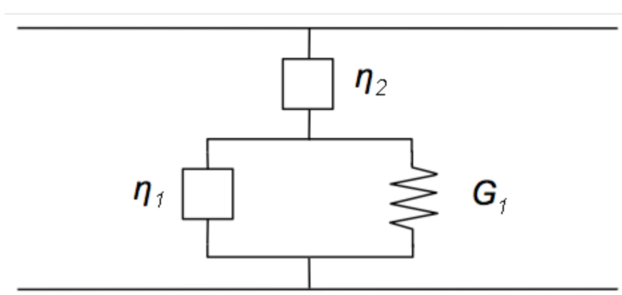
Recall that for a parallel arrangement stresses add $\sigma = \sigma_1 + \sigma_2$ and the strain is equal $\gamma = \gamma_1 = \gamma_2$. Consider the following arrangement:



On the right side we have the relationship $\lambda_1\dot{\sigma}_1 + \sigma_1 = \eta_1\dot{\gamma}_1$ where $\eta_1 = G_1\lambda_1$. On the left side there is the relationship $\sigma_2 = \eta_2\dot{\gamma}_2$.

HW3 Q1: Derive the closed constitutive law for σ, γ, $\dot{\sigma}$, $\dot{\gamma}$, and possibly $\ddot{\gamma}$.

Non-Standard 3-Parameter Voigt Model



$$\sigma_2 = \eta_2\dot{\gamma}_2 \tag{1}$$
$$\sigma_1 = G_1\gamma_1 + \eta_1\dot{\gamma}_1 \tag{2}$$
$$\gamma = \gamma_1 + \gamma_2 \tag{3}$$
$$\dot{\gamma} = \dot{\gamma}_1 + \dot{\gamma}_2 \tag{$\dot{3}$}$$
$$\sigma = \sigma_1 = \sigma_2 \tag{4}$$

Using the above relationships, it follows that:

$$\dot{\gamma}_2 \overset{1}{=} \frac{\sigma_2}{\eta_2} \overset{4}{=} \frac{\sigma}{\eta_2}$$
$$\dot{\gamma} \overset{\dot{3}}{=} \dot{\gamma}_1 + \frac{\sigma}{\eta_2}$$
$$\dot{\gamma} \overset{2,4}{=} \frac{\sigma}{\eta_2} + \frac{1}{\eta_1}\left[\sigma - G_1\gamma_1\right]$$
$$\dot{\gamma} = \left(\frac{1}{\eta_2} + \frac{1}{\eta_1}\right)\sigma - \frac{G_1}{\eta_1}\gamma_1.$$

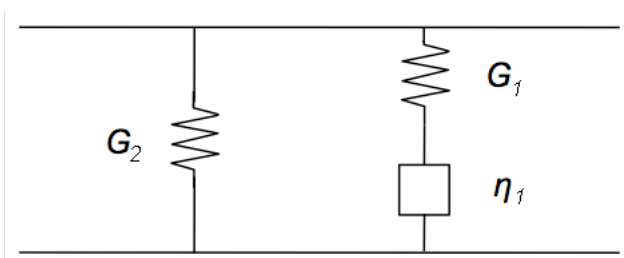
Now, you may try $\gamma_1 \overset{3}{=} \gamma - \gamma_2 \overset{1}{=} \gamma - \int \frac{\sigma}{\eta_2}\, dt$ however this will not work to close the system. Why? Instead we can do the following which involves introducing a second derivative.

$$\eta_1\eta_2\dot{\gamma} = (\eta_1 + \eta_2)\sigma - G_1\eta_2\gamma_1$$
$$\eta_1\eta_2\ddot{\gamma} = (\eta_1 + \eta_2)\dot{\sigma} - G_1\eta_2\dot{\gamma}_1$$
$$\eta_1\eta_2\ddot{\gamma} = (\eta_1 + \eta_2)\dot{\sigma} - G_1\eta_2\left[\dot{\gamma} - \dot{\gamma}_2\right]$$
$$\eta_1\eta_2\ddot{\gamma} = (\eta_1 + \eta_2)\dot{\sigma} - G_1\eta_2\left[\dot{\gamma} - \frac{\sigma}{\eta_2}\right]$$
$$\eta_1\eta_2\ddot{\gamma} + G_1\eta_2\dot{\gamma} = (\eta_1 + \eta_2)\dot{\sigma} + G_1\sigma$$

This gives the linearized scalar Oldroyd B model:

$$\overset{\text{stress}}{G_1\sigma} + \overset{\text{rate of stress}}{(\eta_1 + \eta_2)\dot{\sigma}} = \overset{\text{rate of strain}}{G_1\eta_2\dot{\gamma}} + \overset{\text{rate of rate of strain}}{\eta_1\eta_2\ddot{\gamma}}$$

Standard 3-Parameter Maxwell LVE Solid Model



$$\lambda_1\dot{\sigma} + \sigma = \eta_1\left(1 + \frac{G_2}{G_1}\right)\dot{\gamma} + G_2\gamma$$
$$\dot{\sigma} + \frac{1}{\lambda_1}\sigma = G_1\left(1 + \frac{G_2}{G_1}\right)\dot{\gamma} + \frac{G_2}{\lambda_1}\gamma$$
$$= (G_1 + G_2)\dot{\gamma} + \frac{G_2}{\lambda_1}\gamma$$
$$= (G_1 + G_2)\left[\dot{\gamma} + \frac{G_2}{G_1 + G_2}\frac{1}{\lambda_1}\gamma\right]$$

Where $\lambda_1 = \eta_1/G_1$, $\lambda_2 = \lambda_1 [1 + G_1/G_2]$, and letting $\hat{G} = G_1 + G_2$, the *Canonical Form* of this model is:

$$\dot{\sigma} + \frac{1}{\lambda_1}\sigma = \hat{G}\left[\dot{\gamma} + \frac{1}{\lambda_2}\gamma\right].$$

The Laplace transform of this equation, where $\gamma(0) = \sigma(0) = 0$ is:

$$\tilde{\sigma}(s) = \frac{\hat{G}(s + \lambda_2^{-1})}{s + \lambda_1^{-1}}\tilde{\gamma}(s).$$

HW3 Q2: Use an integrating factor, e^{t/λ_1} to solve for $\sigma(t)$. Show that:

$$\sigma(t) = \left[\sigma(0) - \hat{G}\gamma(0)\right] e^{-t/\lambda_1} + \hat{G}\gamma(t) + \hat{G}(\lambda_2^{-1} - \lambda_1^{-1}) \int_0^t e^{-(t-\tau)/\lambda_1}\gamma(\tau)\, d\tau.$$

If $\gamma(0) = \sigma(0) = 0$, $\sigma(t) = \hat{G}\gamma(t) + m(t) * \gamma(t)$ where $m(t)$ is the memory function $m(t) = \hat{G}(\lambda_2^{-1} - \lambda_1^{-1})e^{-t/\lambda_1}$. Note that since $\lambda_2 > \lambda_1$ the memory function is always negative.

HW3 Q3: Integrate by parts to express the constitutive law in the form:

$$\sigma(t) = \hat{G}\gamma(t) + \text{“}G(t)\text{”} * \dot{\gamma}(t) + \text{boundary term from integration.}$$

We will derive the explicit answer alternatively.

Solution by Convolution from Transform Formula

Lets redo the integrating factor solution as a convolution from transform formula. Recall that $f * g = \mathcal{L}^{-1}(\tilde{f}(s)\tilde{g}(s))$.

$$\begin{aligned}
\tilde{\sigma}(s) &= \frac{\hat{G}(s + \lambda_2^{-1})}{s + \lambda_1^{-1}}\tilde{\gamma}(s) \\
\tilde{\sigma}(s) &= \tilde{M}(s)\tilde{\gamma}(s)
\end{aligned}$$

$$\begin{aligned}
M(t) &= \mathcal{L}^{-1}\left\{\frac{\hat{G}(s + \lambda_2^{-1})}{s + \lambda_1^{-1}}\right\} \\
&= \mathcal{L}^{-1}\left\{\hat{G} + (\lambda_2^{-1} - \lambda_1^{-1})\frac{1}{s + \lambda_1^{-1}}\right\} \\
&= \hat{G}\delta(t) + \hat{G}(\lambda_2^{-1} - \lambda_1^{-1})e^{-t/\lambda_1}H(t)
\end{aligned}$$

$$\sigma(t) = M(t) * \gamma(t)$$

Integration by Parts in Transform Space

$$\begin{aligned}\tilde{\sigma}(s) &= \frac{\hat{G}(s+\lambda_2^{-1})}{s(s+\lambda_1^{-1})} s\tilde{\gamma}(s)\\ \tilde{\sigma}(s) &= \tilde{G}(t)s\tilde{\gamma}(s)\\ \tilde{\sigma}(s) &= \tilde{G}(s)\tilde{\dot{\gamma}}(s)\\ \sigma(t) &= G(t) * \dot{\gamma}(t)\end{aligned}$$

Where the sheer modulus function $G(t)$ is:

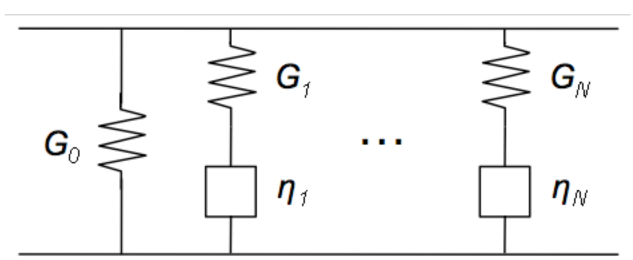
$$\begin{aligned}G(t) &= \mathcal{L}^{-1}\left\{\frac{\hat{G}(s+\lambda_2^{-1})}{s(s+\lambda_1^{-1})}\right\}\\ &= \hat{G}\mathcal{L}^{-1}\left\{\frac{\lambda_1}{\lambda_2}\frac{1}{s} + \left(1-\frac{\lambda_1}{\lambda_2}\right)\frac{1}{s+\lambda_1^{-1}}\right\}\\ &= \hat{G}\frac{\lambda_1}{\lambda_2}\cdot 1 \cdot H(t) + \hat{G}\left(1-\frac{\lambda_1}{\lambda_2}\right)e^{-t/\lambda_1}H(t)\end{aligned}$$

Using the substitutions, $\frac{\lambda_1}{\lambda_2} = \frac{G_2}{G_1+G_2} = \frac{G_2}{\hat{G}}$ and $1-\frac{\lambda_1}{\lambda_2} = \frac{G_1}{G_1+G_2} = \frac{G_1}{\hat{G}}$, and substituting $G(t)$ into $\sigma(t) = G(t) * \dot{\gamma}(t)$ we get:

$$\begin{aligned}\sigma(t) &= \left(G_2H(t) + G_1e^{-t/\lambda_1}H(t)\right) * \dot{\gamma}(t)\\ &= G_2\left[1 * \dot{\gamma}(t)\right] + G_1e^{-t/\lambda_1} * \dot{\gamma}(t)\\ &= G_2\left[\gamma(t) - \gamma(0)\right] + G_1e^{-t/\lambda_1} * \dot{\gamma}(t)\\ &= G_2\left[\gamma(t) - 0\right] + G_1e^{-t/\lambda_1} * \dot{\gamma}(t).\end{aligned}$$

Therefore, $\sigma(t) = G_2\gamma(t) + G_1e^{-t/\lambda_1} * \dot{\gamma}(t)$ where $\lambda_1 G_1 = \eta_1$. This is a superposition formula called the maxwell linear visco-elastic sheer modulus.

N-Mode Maxwell Shear Modulus



$$\begin{aligned}\sigma(t) &= G(t) * \dot{\gamma}(t)\\ G(t) &\equiv G_0H(t) + \sum_{k=1}^{N} G_k e^{-t/\lambda_k}\end{aligned}$$

For "liquids" $G_0 \equiv 0$. The set $\{\lambda_k\}^N$ is the relaxation spectrum.

If I am looking for constitutive laws with sufficient generality to fit data, why not posit:

$$\sigma(t) = G(t) * \dot{\gamma}(t)$$
$$\Downarrow$$
$$\sigma(t) = \int_{-\infty}^{t} G(t-\tau)\dot{\gamma}(\tau)\, d\tau$$

The goal is to use the data "infer" $G(t)$.

Math 769 Lecture 4
1/27/09

Last class we formulated the standard linear visco-elastic (SLVE) solid model solutions for stress relaxation and creep recovery in a slightly modified form.

$$\sigma(t) = G(t) * \dot{\gamma}(t) \Leftrightarrow \tilde{\sigma}(s) = \left[\tilde{G}(s)\right] s\tilde{\gamma}(s)$$
$$\gamma(t) = J(t) * \dot{\sigma}(t) \Leftrightarrow \tilde{\gamma}(s) = [\tilde{J}(s)] s\tilde{\sigma}(s)$$

Where G is the shear modulus function and J is the creep recovery (compliance) function.

Two observations:

1) In step strain, immediate representation: $\sigma(t) = \gamma_0 G(t)$
2) In step stress, immediate representation: $\gamma(t) = \sigma_0 J(t)$

Also: $G(s)J(s) = \frac{1}{s^2} \xrightarrow{\text{Laplace Invesion}} G(t) * J(t) = t$

A basic "mode" of linear elasticity in the Maxwell mode:

$$G(t) = G_0 e^{\frac{-t}{\lambda_0}}$$

Special case of exponential $G(t)$: We can perform all standard experiments analytically.

We have,

$$\sigma(t) = \int_0^t G_0 e^{\frac{-(t-\tilde{t})}{\lambda_0}} \dot{\gamma}(\tilde{t})\, d\tilde{t} \quad , \quad \sigma(0) = 0$$

The general LVE model:

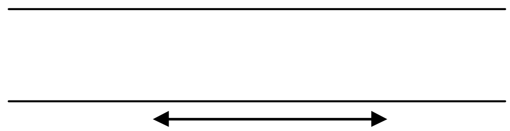
$$\sigma(t) = \int_0^t G_0 e^{\frac{-(t-\tilde{t})}{\lambda_0}} \dot{\gamma}(\tilde{t})\, d\tilde{t}$$

Where $G_0 e^{\frac{-(t-\tilde{t})}{\lambda_0}}$, the special Maxwell shear stress modulus, can be written as $G_{1-\mathrm{Max}}(t-\tilde{t})$.

Oscillatory Shear Experiments:

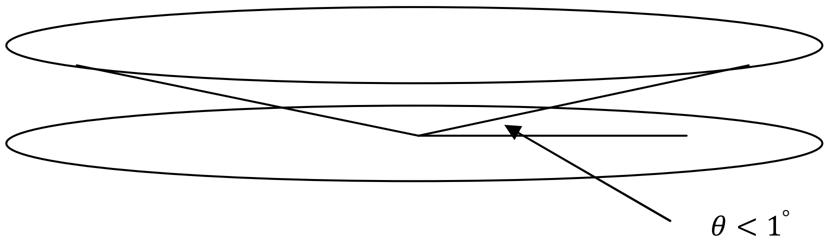
Determine dynamic moduli which transform properties of $G(t)$ or $J(t)$ are explicitly measurable in the lab and therefore we can attain information about some transform of G or J.

Two plates experiment:



Move lower plate

Cone and Plate experiment:



$\theta < 1^{\circ}$

Rotate Plate

Impose oscillatory strain and measure oscillatory stress.

$$\gamma(t) = \gamma_0 \sin(\omega t)$$

$$\dot{\gamma}(t) = \gamma_0 \omega \sin(\omega t)$$

Solve

$$\sigma(t) = \int_{-\infty}^{t} G(t - \tilde{t}) \gamma\omega \cos(\omega \tilde{t}) d\tilde{t}$$

What to expect:

$$\sigma_e = G\gamma \leftrightarrow \text{ stress in-phase with strain}$$

$$\sigma_v = \eta\dot{\gamma} \leftrightarrow \text{ stress out of phase with strain}$$

$$\sigma(t) = (\ A\ ) \sin(\omega t) + (\ B\ ) \cos(\omega t)$$

Where A is how much is in phase and B is how much is out of phase.

Let,

$$s = (t - \tilde{t})$$
$$ds = -d\tilde{t}$$

So the limits of integration change from $-\infty, t$ to $\infty, 0$. Now,

$$\sigma(t) = \int_0^\infty G(s)\gamma_0\omega \cos\big(\omega(t-s)\big)ds$$

$$= \int_0^\infty \gamma_0\omega(G(s)\cos(\omega s)\cos(\omega t))\,ds + \int_0^\infty \gamma_0\omega(G(s)\cos(\omega s)\cos(\omega t))dt$$

$$= \gamma_0\omega\big(\mathcal{F}_c[G](\omega)\big)\cos(\omega t) + \gamma_0\omega\big(\mathcal{F}_s[G](\omega)\big)\sin(\omega t)$$

Where $\mathcal{F}_c[G]$ is the Fourier Cosine Transform of G and $\mathcal{F}_s[G]$ is the Fourier Sine Transform of G.

$$\sigma(t) = \gamma_0 G'(\omega)\sin(\omega t) + \gamma_0 G''(\omega)\cos(\omega t)$$

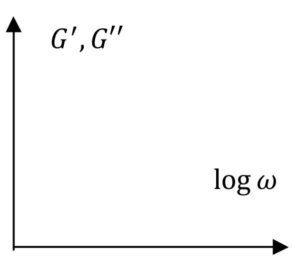
Where $G'(\omega)$ is called the storage modulus and $G''(\omega)$ is called the loss modulus.

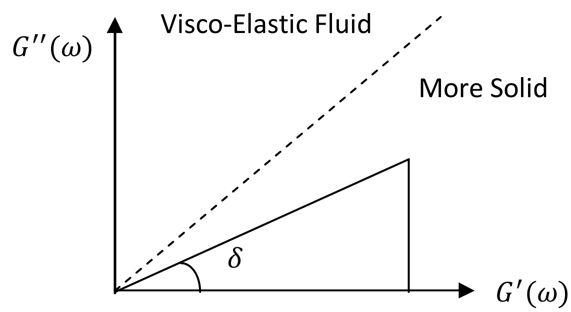
Complex Modulus: $G^*(\omega) = G'(\omega) + iG''(\omega)$

$$G'(\omega) = \omega\mathcal{F}_c[G](\omega)$$

$$G''(\omega) = \omega\mathcal{F}_s[G](\omega)$$

Typical Plots:





Where $\tan\delta = \frac{G'}{G''}$ is called the loss tangent.

Gain some intuition for $G(t)$ approximated by 1 or N-mode Maxwell.

Homework:

Derive the following formulas

$$\mathcal{F}_c\left[G_0 e^{\frac{-t}{\lambda}}\right] = \frac{\lambda G_0}{1 + (\lambda\omega)^2}$$

$$\mathcal{F}_s\left[G_0 e^{\frac{-t}{\lambda}}\right] = \frac{\lambda^2 \omega G_0}{1 + (\lambda\omega)^2}$$

Using the following

$$\lambda \dot{G} + G = 0$$

$$\mathcal{F}_c[\dot{f}] = \omega \mathcal{F}_s[f] - f(0)$$

$$\mathcal{F}_s[\dot{f}] = -\omega \mathcal{F}_c[f]$$

For 1-Mode Maxwell we conclude:

$$G'(\omega) = \omega \mathcal{F}_s[G]$$

$$G' = \frac{(\lambda\omega)^2 G_0}{1 + (\lambda\omega)^2}$$

$$G'' = \frac{\lambda\omega G_0}{1 + (\lambda\omega)^2}$$

Math Modeling II (Math 769)

Lecture 5*

2009.1.27

Review:

$$\sigma(t) = \int_{-\infty}^{t} G(t-\tilde{t})\dot{\gamma}(\tilde{t})d\tilde{t} \tag{1}$$

$$\gamma(t) = \int_{-\infty}^{t} J(t-\tilde{t})\dot{\sigma}(\tilde{t})d\tilde{t} \tag{2}$$

If we impose $\gamma(t) = \gamma_0 \sin(wt)$, from (1) we get:

$$\sigma(t) = \gamma_0 G'(w)\sin(wt) + \gamma_0 G''(w)\cos(wt) \tag{3}$$

With complex shear modulus $G^*(w) = G'(w) + iG''(w)$, we can rewrite (3) as

$$\sigma(t) = \gamma_0 |G^*(w)| \sin(wt+\delta), \;\; \tan(\delta) = \frac{G''}{G'}$$

Alternative derivation:

If we impose $\gamma^* = \gamma_0 e^{iwt}$, $\dot{\gamma}^* = iw\gamma_0 e^{iwt}$, and posit: $\sigma^* = \sigma_0 ei(wt+\sigma)$, then calculate using (1), we will find:

$$\frac{\sigma^*}{\gamma^*} = \frac{\sigma_0}{\gamma_0}e^{i\delta} = G^*(w) \tag{4}$$

Often used alternative V-E moduli:

- G^* - complex shear modulus.
- $\eta^* = \frac{\sigma^*}{\dot{\gamma}^*} = \frac{G^*}{iw} = \eta' - i\eta''$, where $G^* = G' + G''$.

*Notes transcribed by (Bill) Feng Shi

Homework: Starting from (2), analogous to the derivation of $\frac{\sigma^*}{\gamma^*} = G^*(w)$, show that $\frac{\gamma^*}{\sigma^*} = J^*(w)$, and $G^*J^* = 1$.

Example of tensor equation:

Component form: $\sigma_{xy}(t) = \int_{-\infty}^{t} G(t-\tilde{t})\dot{\gamma}_{xy}(\tilde{t})d\tilde{t}$.

Matrix form: $\underline{\underline{\sigma}}(t) = \int_{-\infty}^{t} G(t-\tilde{t})\underline{\underline{D}}(\tilde{t})d\tilde{t}$,

where $\underline{\underline{D}} = (\nabla \underline{V} + \nabla \underline{V}^t)/2$, $\underline{V} = \dot{\gamma}(t)(y,0,0)$, e.g. the velocity field of the linear shear flow.

Tensor notations:

$\underline{\mathrm{V}}$ — vector (rank 1 tensor),
$\underline{\underline{\mathrm{V}}}$ — 3 by 3 matrix (rank 2 tensor).

Cartesian tensors:

let $\{\underline{e}_i\}$, $\{\underline{e}'_j\}$ be two orthonormal frames, i.e. $\underline{e}_i \cdot \underline{e}_j = \delta_{ij} = \underline{e}'_i \cdot \underline{e}'_j$. Then a vector $\underline{\mathrm{v}}$ can be represented in these frames as:

$$\underline{v} = \sum(\underline{v} \cdot \underline{e}_i)\underline{e}_i = \sum v_i \underline{e}_i$$
$$= \sum(\underline{v} \cdot \underline{e}'_i)\underline{e}'_i = \sum v'_i \underline{e}'_i$$

where $v_i = |\underline{v}| \cos \angle(\underline{v}, \underline{e}_i)$, $v'_i = |\underline{v}| \cos \angle(\underline{v}, \underline{e}'_i)$.

A transformation of coordinates in R^3 between $\{\underline{e}_i\}$ and $\{\underline{e}'_i\}$ is given by:

$$\underline{e}_i = \sum_j \cos \angle(\underline{e}_i, \underline{e}'_j)\underline{e}'_j = \sum_j l_{ij}\underline{e}'_j$$

$$\underline{e}'_j = \sum_i \cos \angle(\underline{e}'_j, \underline{e}_i)\underline{e}_i = \sum_i l_{ij}\underline{e}_i$$

Define the transformation matrix L and $\mathcal{L}$ as:

$$(L)_{ij} = (\mathcal{L}^t)_{ij} = l_{ji} \ or \ (\mathcal{L})_{ij} = (L^t)_{ij} = l_{ij}$$

It is easy to see that: $\underline{\underline{\mathcal{L}}} = \underline{\underline{L}}^t, \underline{\underline{L}}\,\underline{\underline{L}}^t = \underline{\underline{\mathcal{L}}}\,\underline{\underline{\mathcal{L}}}^t = I$.

Then any vector $\underline{\mathrm{v}}$ in one Cartesian frame $\{\underline{e}_i\}$ transforms to $\underline{v}'$ in any other frame $\{\underline{e}'_i\}$ by the relations:

$$\underline{v}' = \underline{\underline{L}}\,\underline{v} = \underline{\underline{\mathcal{L}}}^t\,\underline{v}, \quad \underline{v} = \underline{\underline{\mathcal{L}}}\,\underline{v}'$$

Math Modeling II (Math 769)

Lecture 6*

2009.2.3

Review

Last class we derived the change of orthonormal frames in Cartesian coordinates. If we take $\{\underline{e}_i\}$, $\{\underline{e}_i'\}$ to be two orthonormal frames, then one way to describe the change of coordinates is:

$$\underline{e}_i = \sum_j l_{ij}\underline{e}_j', \quad \underline{e}_j' = \sum_i l_{ij}\underline{e}_i$$

The other is to use matrix notation. Define the transform matrix as:

$$(\underline{\underline{L}})_{ij} = (\underline{\underline{\mathcal{L}}}^t)_{ij} = l_{ji} \; or \; (\underline{\underline{\mathcal{L}}})_{ij} = (\underline{\underline{L}}^t)_{ij} = l_{ij}$$

where $l_{ij} = \cos\angle(\underline{e}_i, \underline{e}_j')$.

Homework Q1: Discuss the geometric meaning why $\underline{\underline{L}}^t = \underline{\underline{L}}^{-1}$.

A tensor in R^3 of rank 1 inherits the properties of coordinate changes, i.e. 3 scalars $\{x_i\}_{i=1}^3$ which transform under change of coordinates by this rule: $\underline{x} \to \underline{x}' = \underline{\underline{L}}\,\underline{x}$, where L is a pure rotation is a special coordinate change.

Broad classes of coordinates change

- Special geometries (polar, cylindrical, ellipsoidal...);
- Deformations of flows:
  $x \to x' = f(x)$, where $f(x)$ can be any kind of nonlinear functions such as flow map with nonsingular linearization(det $\frac{df}{dx} \neq 0$).

*Notes are transcribed by (Bill) Feng Shi and edited by Gregory Forest

Example: polar coordinates in R^2 or cylindrical coordinates in R^3 are:

$$x'_1 = \lambda = g_1 = \sqrt{x_1^2 + x_2^2}$$
$$x'_2 = \phi = g_2 = \tan^{-1} \tfrac{x_2}{x_1}$$
$$x'_3 = x_3 = g_3 = x_3$$

Rank of a tensor:

- Scalar(rank 0) are independent of coordinates;
- A tensor of rank 1 has 3 components(scalars) ξ_i in coordinates $\{\underline{x}_i\}$, or 3 components ξ'_i in coordinates $\{\underline{x}'_i\}$, which obey $\underline{\xi}' = \underline{\underline{L}}\,\underline{\xi}$, $\underline{\xi} = \underline{\underline{\mathcal{L}}}\,\underline{\xi}'$; or $\xi'_j = \sum l_{ij}\xi_i$, $\xi_j = \sum l_{ij}\xi'_i$.
- A tensor of rank k has 3^k components which transform under a Cartesian coordinate change by k products of l_{ij}.
  Example: k=2. We have a rank 2 tensor $\underline{\underline{\tau}}$ whose components are τ_{ij} in coordinates x_k, and τ'_{ij} in coordinates x'_k, which obey:

$$\tau'_{ij}(\underline{x}') = \tau_{mn}(\underline{x}) l_{mi} l_{nj} \tag{1}$$

$$\tau ij(\underline{x}) = \tau'_{mn}(\underline{x}') l_{im} l_{jn} \tag{2}$$

  Homework Q2: Confirm that equations (1)(2) are equivalent to

$$\underline{\underline{\tau}}' = \underline{\underline{L}}\,\underline{\underline{\tau}}\,\underline{\underline{L}}^t \tag{3}$$

$$\underline{\underline{\tau}} = \underline{\underline{L}}^t\,\underline{\underline{\tau}}\,\underline{\underline{L}} \tag{4}$$

- Vector fields are tensors of rank 1 arising from gradients of scalar fields.
- Rank 2 tensor fields arise from gradients of rank 1 tensor fields.

Example:

The gradient field of a scalar field(temperature, density, one velocity component) $f(\underline{x}, t)$ is $(\nabla_{\underline{x}} f)_i = \frac{\partial f}{\partial x_i}$, $i = 1, 2, 3$.

Suppose we take the change of coordinates: $x \to x'$. Then by chain rule, the vector field is transformed to $\frac{\partial f}{\partial x'_i} = \sum_j \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial x'_i}$, or in a more compact form, $\frac{\partial f}{\partial \underline{x}'} = \frac{\partial f}{\partial \underline{x}}\frac{\partial \underline{x}}{\partial \underline{x}'}$, or $\frac{\partial f}{\partial \underline{x}'} = \frac{\partial \underline{x}}{\partial \underline{x}'}\frac{\partial f}{\partial \underline{x}}$ (depending on whether you have defined the gradient of a scalar as a row or column).

Differentials: $\underline{x}$, $d\underline{x} \to d\underline{x}' = \frac{\partial \underline{x}'}{\partial \underline{x}} d\underline{x}$ or $d\underline{x}\frac{\partial \underline{x}'}{\partial \underline{x}}$ (depending on whether you are defining row or column vectors for coordinates.).

Special rank 2 tensors:

Suppose $\underline{v}$ is any vector field (i.e. any rank 1 tensor field); then the gradient of $\underline{v}$ is $(\nabla_{\underline{x}}\underline{v})_{ij} = \frac{\partial v_i}{\partial x_j}$ or $\frac{\partial v_j}{\partial x_i}$ (depending on whether $\underline{v}$ is a row or column vector).

The symmetric part of $\nabla_{\underline{x}}\underline{v}$ is $\frac{1}{2}[\nabla_{\underline{x}}\underline{v} + (\nabla_{\underline{x}}\underline{v})^t] \equiv \underline{\underline{D}}$, which is called the rate-of-strain tensor if $\underline{v}$ is a velocity field.

The anti-symmetric part of $\nabla_{\underline{x}}\underline{v}$ is $\frac{1}{2}[\nabla_{\underline{x}}\underline{v} - (\nabla_{\underline{x}}\underline{v})^t] \equiv \underline{\underline{\Omega}}$, which is called the vorticity tensor if $\underline{v}$ is a flow field.

Special rank 2 tensors:

How does arc length change in any (Cartesian) coordinates change?

An infinitesimal arc length is defined as $ds^2 = d\underline{x} \cdot d\underline{x} =< d\underline{x}, d\underline{x} >= \sum dx_i^2$; now if we change the coordinates $\underline{x} \to \underline{x}'$, then $dx_i = \sum_j \frac{\partial x_i}{\partial x'_j} dx'_j$, thus

$$ds^2 = \sum_i (\sum_j \frac{\partial x_i}{\partial x'_j} dx'_j)^2 = \sum_i \sum_j g_{ij} dx'_i dx'_j$$

where g_{ij} is called the metric tensor.(We will return to this topic soon when we discuss deformations.)

Homework Q3: Let g_{ij} be the metric tensor introduced above.

(a). Show that $g_{ij} = \sum_k \frac{\partial x_k}{\partial x'_i} \frac{\partial x_i}{\partial x'_j}$.

(b). For Cartesian maps: $\underline{x}' = \underline{\underline{L}}\,\underline{x}$, $\underline{\underline{L}}^t = \underline{\underline{L}}^{-1}$, show that $g_{ij} = \delta_{ij}$

(c). For cylindrical coordinates: $x'_1 = \sqrt{x_1^2 + x_2^2}$, $x'_2 = \tan^{-1}(\frac{x_2}{x_1})$, $x'_3 = x_3$, show that

$$g_{ij} = \begin{cases} 1 & \text{if } i = j = 1, 3 \\ (x'_1)^2 & \text{if } i = j = 2 \\ 0 & \text{otherwise} \end{cases}$$

(d). For spherical coordinates: $x'_1 = \eta = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $x'_2 = \theta = \tan^{-1}(\frac{x_2}{x_1})$,

$x'_3 = \phi = \tan^{-1}(\frac{\sqrt{x_1^2+x_2^2}}{x_3})$, show that

$$g_{ij} = \begin{cases} 1 & \text{if } i = j = 1 \\ \eta^2 \sin\phi & \text{if } i = j = 2 \\ \eta^2 & \text{if } i = j = 3 \\ 0 & \text{otherwise} \end{cases}$$

Definition: Coordinates are orthogonal if the metric tensor is diagonal. And $g_{ij} = h_i^2$ are called the "scale factors", where h_i are the diagonal entries.

$$ds^2|_{\text{restricted to direction of } x'_i \text{ while others are held constant}} = h_i^2(dx'_i)^2$$

Homework Q4: For cases of cylindrical, spherical and Cartesian coordinates, what are the relationships among level surfaces defined by x'_i= constant, and then for any orthogonal coordinate system?

Fundamental theorem of linear algebra: Quotient rule

State for rank 2 tensor first:

If A_{ij} are 9 scalars, independent of $\underline{b}$, a rank 1 tensor(vector), and $\underline{c}$ is any vector(rank 1 tensor) such that A_{ij} satisfy:

$$\sum_j A_{ij} b_j = c_i \iff \underline{\underline{A}}\, \underline{b} = \underline{c}$$

then $\underline{\underline{A}}$ is a rank 2 tensor.

Homework Q5: Prove that under the orthogonal transform $\underline{b}' = \underline{\underline{L}}\, \underline{b}$, $\underline{c}' = \underline{\underline{L}}\, \underline{c}$, $\underline{\underline{A}}$ is transformed to $\underline{\underline{A}}' = \underline{\underline{L}}\, \underline{\underline{A}}\, \underline{\underline{L}}^t$

Lecture Notes - Math 769
Crash Course in Continuum Mechanics continued

February 5, 2009

In the last class we discussed the following relationships for stress and strain:

$$\text{(liquids) } \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{C}}}}\ \underline{\underline{\gamma}}$$

$$\text{(solids) } \underline{\underline{\sigma}} = \underline{\underline{\underline{\underline{\eta}}}}\ \underline{\underline{\mathbb{D}}}$$

Where $\mathbb{D}$ is the rate of strain tensor. We note that this leads to 3^4 viscosities to describe the relationship between stress and strain in the second equation thus we turn our attention to isotropic tensors to reduce the number of parameters needed.

Definition: An *isotropic tensor* is a tensor whose components are invariant under coordinate changes.

We can find the dimension of the set of isotropic tensors in different ranks as follows :

- **rank 0** : By definition all tensors in this set are isotropic, i.e. $\dim\left\{ \substack{\text{rank 1} \\ \text{iso}} \right\} = 1$
- **rank 1** : Here only $\underline{0}$ is isotropic. Thus we say that $\dim\left\{ \substack{\text{rank 1} \\ \text{iso}} \right\} = 0$
- **rank 2** : The only rank 2 tensor that is isotropic properties is $\underline{\underline{\mathbf{I}}} = \delta_{i,j}$ Note we easily see that $\underline{\underline{\mathbf{I}}} = \underline{\underline{\mathrm{L}}}^t\ \underline{\underline{\mathbf{I}}}\ \underline{\underline{\mathrm{L}}}$, which gives this result and that $\dim\left\{ \substack{\text{rank 2} \\ \text{iso}} \right\} = 1$
- **rank 3** : The rank 3 tensor defined by ϵ_{ijk} is the only isotropic tensor in this class also giving $\dim\left\{ \substack{\text{rank 3} \\ \text{iso}} \right\} = 1$
- **rank 4** : Here the following basis gives that $\dim\left\{ \substack{\text{rank 4} \\ \text{iso}} \right\} = 1$: $\{ \delta_{ij}\delta_{kl}, \delta_{ik}\delta_{jl} \pm \delta_{il}\delta_{jk} \}$. Taking only the symmetric part, $\{ \delta_{ij}\delta_{kl}, \delta_{ik}\delta_{jl} \pm \delta_{il}\delta_{jk} \}$ of this basis gives $\dim\left\{ \substack{\text{rank 4} \\ \text{iso, sym}} \right\} = 2$. This is relevant for our purposes since it is the reason we classify material with two individual types of viscosities (shear and dilatation)and two types of moduli (shear and Youngs).

For an isotropic solid if we restrict $\underline{\underline{\underline{\underline{\mathrm{C}}}}}$ to the basis $\{ \delta_{ij}\delta_{kl}, \delta_{ik}\delta_{jl} \pm \delta_{il}\delta_{jk} \}$ this yields **Hookes law** :

$$\sigma_{i,j} = (\lambda e_{k,k})\delta_{ij} + 2\mu\gamma_{i,j}$$

where $\lambda e_{k,k}$ is the Youngs moduli and 2 μ is the shear modulus.

Similarly for isotropic liquids we have **Stokes Law** :

$$\sigma_{i,j} = -\mathrm{p}\delta_{ij} + 2\mu\mathbb{D}_{i,j} + \lambda(tr\mathbb{D})\delta_{ij}$$

Where is the μ shear viscosity and is the λ dilation viscosity

Tensor operations and operators

Dyadic products: The unit coordinate vectors are defined by the set of unit dyads : $e_i \equiv \delta_i$. We define the dyadic product as :

$$\underline{e}_i \otimes \underline{e}_j = \begin{cases} \text{1 in entry i, j} \\ \text{0 otherwise} \\ \text{in a 3x3 matrix} \end{cases}$$

and the unit dyadic products generate a basis of rank 2 tensors. For example, $\underline{\underline{\tau}}$ can be written as

$$\underline{\underline{\tau}} = \sum_{i,j} \tau_{i,j} \;\; \underline{e}_i \otimes \underline{e}_j$$

Tensor operations: . With the definition of Dyadic products established we move on to tensor operations. Note that we will define these on the unit dyads since it is easily generalized to all rank 2 tensors by linearity. Further reading and more detailed explanations can be found in Bird, Armstrong and Hassogen, Vol 1 Appendix.

- Contractions:
  - The symbol " $\cdot$ " is a contraction which maps two rank 2 tensors to a rank 1 tensor.
  - The symbol " **:** " is called a double contraction which maps two rank 2 tensors to a rank 0 tensor.
- Norm of a Matrix:

$$(\underline{e}_i \otimes \underline{e}_j)\mathbf{:}(\underline{e}_k \otimes \underline{e}_l) \equiv (\underline{e}_i \cdot \underline{e}_j) \overset{\text{(times)}}{\text{x}} (\underline{e}_i \cdot \underline{e}_j) = \delta_{ij}\delta_{lk}$$

- Matrix vector product (left):

$$\underline{\underline{A}} \cdot \underline{b} = (\underline{e}_i \otimes \underline{e}_j) \cdot \underline{e}_k = \underline{e}_i(\underline{e}_j \cdot \underline{e}_k) = \delta_{jk}\underline{e}_i$$

- Matrix vector product (right):

$$\underline{b} \cdot \underline{\underline{A}} = \underline{e}_i \cdot (\underline{e}_j \otimes \underline{e}_k) = (\underline{e}_i \cdot \underline{e}_j)\underline{e}_k = \delta_{ij}\underline{e}_k$$

- Matrix matrix product :

$$\underline{\underline{A}}\;\underline{\underline{B}} = (\underline{e}_i \otimes \underline{e}_j)(\underline{e}_k \otimes \underline{e}_l) = (\underline{e}_j \cdot \underline{e}_k)\underline{e}_i \otimes \underline{e}_l = \delta_{j,k}(\underline{e}_i \otimes \underline{e}_l)$$

- Matrix vector cross product (left):

$$\underline{\underline{A}} \times \underline{b} = (\underline{e}_i \otimes \underline{e}_j) \times \underline{e}_k = \underline{e}_i(\underline{e}_j \times \underline{e}_k) = \sum_l \epsilon_{jkl}(\underline{e}_i \otimes \underline{e}_l)$$

- Matrix vector cross product (right):

$$\underline{b} \times \underline{\underline{A}} = \underline{e}_l(\underline{e}_j \otimes \underline{e}_k) = \sum_l \epsilon_{ijl}(\underline{e}_l \otimes \underline{e}_k)$$

- Special symmetries of rank 2 tensors:
  - symmetric : $\tau_{i,j} = \tau_{j,i}$
  - antisymmetric : $\tau_{i,j} = -\tau_{j,i}$
  - transpose : $\tau_{i,j} \to \tau_{j,i}$
- Dyadic product of two vectors :

$$\underline{v} \otimes \underline{w} = \sum_{i,j} v_i w_j (\underline{e}_i \otimes \underline{e}_j)$$

- Norm of τ :

$$\|\tau\| = \sqrt{\frac{1}{2}\tau\colon\tau^t} = \sqrt{\frac{1}{2}\sum_{i,j}\tau_{i,j}^2}$$

- Scalar product:

$$\underline{\underline{\sigma}}\colon\underline{\underline{\tau}} \equiv \sum_i \sum_j \sigma_{i,j}\tau_{j,i}$$

- Tensor product:

$$\underline{\underline{\sigma}} \;\; \underline{\underline{\tau}} = \sum_i \sum_l \underline{e}_i \otimes \underline{e}_l \left(\sum_j \sigma_{i,j}\tau_{j,l} \right)$$

Special features of tensors

- Invariants of a rank 2 tensor:
  Here we are referring to being invariant under orthogonal coordinate changes.
  - The trace of the matrix, $\mathrm{tr}(\tau) = \sum \tau_{i,i}$, as well as $\mathrm{tr}(\tau^2)$ and $tr(\tau^3)$
  - The characteristic polynomial is defined by :

$$\begin{aligned} 0 &= \det(\lambda\underline{\underline{I}} - \underline{\underline{\tau}}) \\ &= \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 \\ I_1 &= \mathrm{tr}(\tau) \\ I_2 &= \frac{1}{2}((\mathrm{tr}(\tau))^2 - \mathrm{tr}(\tau^2)) \\ I_3 &= \frac{1}{6}((\mathrm{tr}(\tau))^3 - 3\mathrm{tr}(\tau)\mathrm{tr}(\tau^2) + 2\mathrm{tr}(\tau^3)) \end{aligned}$$

  where each of the I_i's are invariant which follows from the properties of the trace. This naturally leads us to the following theorem whose importance will be apparent later on.

Cayley - Hamilton Theorem: Every rank 2 tensor satisfies its own characteristic polynomial. i.e. for our example $\underline{\underline{\tau}}^3 - I_1\underline{\underline{\tau}}^2 + I_2\underline{\underline{\tau}} - I_3 = 0$

Math 769 Notes from Feb. 10, 2009

Jessica wehner

Some terminology and definitions:

Gradient: $\underline{\nabla} = \underline{e}_1 \dfrac{d}{d\underline{x}_1} + \ldots = \Sigma \underline{e}_i \dfrac{d}{d\underline{x}_i}$.

Divergence: $\underline{\nabla} \cdot \underline{v} = (\Sigma \underline{e}_i \dfrac{d}{d\underline{x}_i})(\Sigma \underline{e}_j \underline{v}_j) = \Sigma \dfrac{d\underline{v}_i}{d\underline{x}_i} = tr(\underline{\nabla}_{\underline{x}} \underline{v})$.

Curl of rank 1 tensor: $\underline{\nabla} \times \underline{v}$ written as $| \cdot |$ or ϵ_{ijh}.

note: $\underline{\nabla}\ \underline{v} = (\underline{\nabla}\ \underline{v}_1\ \underline{\nabla}\ \underline{v}_2\ \underline{\nabla}\ \underline{v}_3)$.

Divergence of rank 2 tensor (converts to rank 1): $div\ \underline{\underline{\tau}} = \underline{\nabla} \cdot \underline{\underline{\tau}} = \sum_k \underline{e}_k (\sum_i \dfrac{d\underline{\underline{\tau}}_{ik}}{d\underline{x}_i})$.

note: $(div \underline{\underline{\tau}})_j = \sum_i \dfrac{d\underline{\underline{\tau}}_{ij}}{d\underline{x}_i}$.

$**$Reference Bird, Armstrong and Hassager, vol.1 Appendix A, as well as Tables 3.1-3.3.$**$

$v(x)$ is irrotational if $\underline{\nabla} \times \underline{x} = \underline{0}$ (curl free)
$div\ \underline{v} = \underline{0}$

Be sure to understand how to do Coordinate changes in Orthogonal Bases!
Ex: Write $\underline{\nabla}$ in any orthogonal coordinate system.

Ex: Cylindrical $\underline{\delta}_r = \cos\theta \underline{\delta} x + \sin\theta \underline{\delta} y + 0 \cdot \underline{\delta} z$

$\underline{\delta}_\theta = -\sin\theta \underline{\delta} x + \cos\theta \underline{\delta} y + 0 \cdot \underline{\delta} z$

$\underline{\delta} z = \underline{\delta} z$

Then $\underline{\nabla} \equiv \delta_{\underline{x}} \dfrac{d}{d\underline{x}} + \ldots$ becomes $\underline{\delta}_r \dfrac{d}{dr} + \underline{\delta}_\theta \dfrac{1}{r} \dfrac{d}{d\theta} + \underline{\delta}_z \dfrac{d}{dz}$.

Deformation and Flow

Biomechanics: reference text by Fung

Measures of strain: elongational deformation vs. shear deformation

Elongation

Elongational deformation occurs, for example, when a sample is stretched lengthwise from a length L_0 to a final length L. There are a few ways to measure this.
Linear Measure:

$$\lambda = \frac{L}{L_0} = \quad \text{stretch ratio}$$

$$\epsilon' = \frac{L - L_0}{L} \rightarrow 1 - \lambda^{-1}$$

$$\epsilon = \frac{L - L_0}{L_0} \rightarrow \lambda - 1$$

Nonlinear Measure:

$$e = \frac{L^2 - L_0^2}{2L^2} = \left[\frac{L - L_0}{L}\right]\left[\frac{L + L_0}{2L}\right]$$

$$\text{or} \quad \varepsilon = \frac{L^2 - L_0^2}{2L_0^2} = \left[\frac{L - L_0}{L_0}\right]\left[\frac{L + L_0}{2L_0}\right]$$

Note that if $L \sim L_0$, then e and ε are about the same up to $\mathrm{O}(\lambda)$.

Shear

Shear deformation occurs, for examples, when a sample is trapped between two plates and the top or bottom plate is slid a distance x_0 while the other is held in place. The original length of the edge of the sample (and also the length between the plates) is L_0, and as the plates slide, this edge is rotated through an angle α and to a new length $L_0 \rightarrow L_0\sqrt{1 + \left(\frac{x_0}{L_0}\right)^2} = L_0\sqrt{1 + \tan^2\alpha}$.

Now consider that at time $t = 0$ you have a point $\underline{\xi}$ in a reference configuration, and at time t the point is $\underline{x}(t, \underline{\xi})$ in the present configuration. Then $\underline{x}(\underline{\xi})$ is a deformation. Also, $\underline{\underline{F}} \equiv \frac{d\underline{x}}{d\underline{\xi}}$ is a deformation gradient (rank 2 tensor).

In Flows: Focus on particle paths.

$$\left\{ \begin{array}{l} \dfrac{d\underline{x}}{dt} = \underline{v}(\underline{x}, t) \\ \underline{x}(0) = \underline{x}_0 = \underline{\xi} \end{array} \right. \rightarrow \underline{x}(t, \underline{\xi})$$

Be careful about whether $\underline{x}$ and $\underline{v}$ represent rows or columns!

$$\underline{\underline{F}} = \frac{d\underline{x}}{d\underline{\xi}} \text{ and } \frac{d\underline{v}}{d\underline{x}} = \nabla_{\underline{x}}\underline{v}$$

Rate of change of $\underline{\underline{F}}$:

$$\frac{d}{dt}\underline{\underline{F}} = \frac{d}{dt}\left(\frac{d\underline{x}}{d\underline{\xi}}\right) = \frac{d\frac{d\underline{x}}{dt}}{d\underline{\xi}} = \frac{d\underline{v}}{d\underline{\xi}} = \frac{d\underline{v}}{d\underline{x}}\frac{d\underline{x}}{d\underline{\xi}} = \frac{d\underline{v}}{d\underline{x}}\underline{\underline{F}}$$

So,

$$\frac{d}{dt}\underline{\underline{F}} = (\underline{\nabla}\ \underline{v})\underline{\underline{F}}$$

$$\dot{\underline{\underline{F}}}\ \underline{\underline{F}}^{-1} = \nabla\underline{v}$$

These relations do not depend on the configuration, so we would like to use these to build our constitutive laws.

Since $\underline{\underline{F}}$ is invertible, it has two polar decompositions:

$$\underline{\underline{F}} = \underline{\underline{V}} \cdot \underline{\underline{R}} = \underline{\underline{R}} \cdot \underline{\underline{U}}$$

HWK #7: Review Polar Decomposition (from linear algebra) and indicate how the eigenvectors and eigenvalues of $\underline{\underline{F}}$ determine $\underline{\underline{U}}$, $\underline{\underline{V}}$, and $\underline{\underline{R}}$.

- $\underline{\underline{R}}$ is a pure rotation.
- $\underline{\underline{U}}$ and $\underline{\underline{V}}$ are dilations $\rightarrow$ poitive, definite and symmetric.
- $\underline{\underline{V}}\ \underline{\underline{R}} = \underline{\underline{R}}\ \underline{\underline{U}}$ implies $\underline{\underline{U}} = \underline{\underline{R}}^T\ \underline{\underline{V}}\ \underline{\underline{R}}$ and $\underline{\underline{V}} = \underline{\underline{R}}\ \underline{\underline{U}}\ \underline{\underline{R}}^T$.

We let $\underline{\underline{F}} = \underline{\underline{R}}\ \underline{\underline{U}}$ and proceed to remove the dependence on $\underline{\underline{R}}$:

$$\underline{\underline{F}} = \underline{\underline{R}}\ \underline{\underline{U}}$$

$$\underline{\underline{F}}^T = \underline{\underline{U}}^T\ \underline{\underline{R}}^T = \underline{\underline{U}}^T\underline{\underline{R}}^{-1}$$

$$\underline{\underline{F}}^T\ \underline{\underline{F}} = \underline{\underline{U}}^T\ \underline{\underline{R}}^{-1}\ \underline{\underline{R}}\ \underline{\underline{U}} = \underline{\underline{U}}^T\ \underline{\underline{U}} = \underline{\underline{U}}^2 = \underline{\underline{C}}, \text{ the Cauchy-Green tensor.}$$

$$\underline{\underline{F}} = \underline{\underline{V}}\ \underline{\underline{R}}$$

$$\underline{\underline{F}}^T = \underline{\underline{R}}^T\ \underline{\underline{V}}^T = \underline{\underline{R}}^{-1}\ \underline{\underline{V}}^T$$

$$\underline{\underline{F}}\ \underline{\underline{F}}^T = \underline{\underline{V}}\ \underline{\underline{R}}\ \underline{\underline{R}}^{-1}\ \underline{\underline{V}}^T = \underline{\underline{V}}\ \underline{\underline{V}}^T = \underline{\underline{V}}^2 = \underline{\underline{B}}, \text{ the Finger tensor.}$$

(Observe that these equations are the tensor analogs of e and ε from elongation.) Now, since $\underline{\underline{F}} = \frac{d\underline{x}}{d\underline{\xi}}$ maps from reference to present, and $\underline{\underline{F}}^T$ maps from present to reference, then $\underline{\underline{C}}$ maps from reference to reference and $\underline{\underline{B}}$ maps from present to present, so both $\underline{\underline{C}}$ and $\underline{\underline{B}}$ are coordinate-free (dimensionless).

Now we may posit that stress is proportional to $\underline{\underline{C}}$ or $\underline{\underline{B}}$, and invarient under orthogonal transformations, which forms a stress/strain constitutive law which does not depend on the coordinate system in use.

$**$Note that these laws had to become nonlinear so that we didn't induce stress by simply rotating the system.

MATH769-Lecture

Shengqian Chen

February 24, 2009

In this class, we are going to discuss three canonical homogenous flow. The initial position of a fluid particle is $\underline{\xi}$. After time t, the new location is $\underline{x}(\underline{\xi}, t)$ which satisfies the initial condition $\underline{x}(\underline{\xi}, 0) = \underline{\xi}$.

1 Extensional flow

The velocity ~~of the~~ extensional flow is given by

$$\underline{v} = \begin{pmatrix} \dot{\epsilon}_1(t) & 0 & 0 \\ 0 & \dot{\epsilon}_2(t) & 0 \\ 0 & 0 & \dot{\epsilon}_3(t) \end{pmatrix} \underline{x} \equiv \underline{\underline{L}}(t)\underline{x}$$

where $\dot{\epsilon}(t)$ are extensional rates. This is equivalent to

$$\begin{aligned} v_{x_1} &= \dot{v}_1 = \dot{\epsilon}_1(t)x_1 \\ v_{x_2} &= \dot{v}_2 = \dot{\epsilon}_2(t)x_2 \\ v_{x_3} &= \dot{v}_3 = \dot{\epsilon}_3(t)x_3 \end{aligned} \tag{1}$$

Since $\underline{v} = \dot{\underline{x}}$, this is actually an ODE system written as $\dot{x}_j = \dot{\epsilon}(t)x_j$. When $\dot{\epsilon}_j(t)$ are constant, the position of the particle can be solved by integration $x_j(t) = x_j(0)e^{\dot{\epsilon}t}$. In general, however, $\dot{\epsilon}_j$ are time dependent then

$$x_j(t) = x_j(0)e^{\int_0^t \dot{\epsilon}_j(s)ds}$$

Remarks

1. At any time t^*, we have

$$\underline{x}(t^*) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \underline{x}(0)$$

where $\alpha_j = e^{\dot{\epsilon}_j(t^*)}$ when $\dot{\epsilon}_j$ are constant, so for volume pressing (incompressible) flow,

$$div(\underline{v}) = tr(\underline{\underline{L}}) \equiv \sum \dot{\epsilon}_j = 0$$

and

$$\prod \alpha_j = e^{\sum \dot{\epsilon}_j t^*} = 1$$

2. For the incompressible, uniaxial homogeneous flow with extension along $\cancel{l}$ axis, the fact that

$$\dot{\epsilon}_1 > 0, \dot{\epsilon}_2 = \dot{\epsilon}_3 = -\frac{1}{2}\dot{\epsilon}_1$$

recovers

$$\alpha_1 > 1, \alpha_2 = \alpha_3 = \frac{1}{\sqrt{\alpha_1}}$$

~~in deformation.~~

3. When the extensional flow has ~~the~~ constant rate, i.e. $\dot{\epsilon}_j$ = constant, the extensional flow is the prototype of strong flow because

$$\frac{\partial x_j(t)}{\partial x_j(0)} = e^{\dot{\epsilon}_j(t)}$$

Note It is easy to find the special velocity gradient tensors for the uniaxial, incompressible, extensional flow with constant rate $\dot{\epsilon}(t)$. Since

$$\underline{v} = \dot{\epsilon}_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \underline{x}$$

we have

$$\frac{\cancel{\underline{v}}}{\cancel{\underline{x}}} = \dot{\epsilon}_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

therefore $\underline{\underline{D}} = \frac{\partial v}{\partial \underline{x}}$ and $\underline{\underline{\Omega}} = \underline{\underline{0}}$, indicating the flow has no rotation.

2 Rotation flow

The second canonical homogenous flow is rotation ~~along~~ x_3 axis.

$$\underline{v} = \underline{\dot{x}} = \begin{pmatrix} 0 & \omega(t) & 0 \\ -\omega(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} \equiv \underline{\underline{\Omega}}(t)\underline{x}$$

where $\omega(t)$ is constant or time-dependent, then $\underline{\underline{D}} = \underline{\underline{0}}$.

HOMEWORK Derive the particle trajectory formula for $\omega(t)$ ~~is~~ constant or time-dependent function. Then recover the rotation deformation

$$\underline{x}(t) = \underline{\underline{\Omega}}(t)\underline{x}(0), \tag{2}$$

and relate $\underline{\underline{\Omega}}$ with $\underline{\underline{F}}$ in deformation.

3 Simple Shear

The third canonical homogenous flow is simple shear~~, or, linear shear.~~

$$\underline{v} = \dot{\gamma}(t)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\underline{x}$$

Substitute $\dot{\underline{x}} = \underline{v}$ and solve the ODE system then get the particle trajectory equations when $\dot{\gamma}(t)$ is constant,

$$\begin{aligned} x_3 &= x_3(0) \\ x_2 &= x_2(0) \\ x_1 &= \dot{\gamma}x_2(0)t + x_1(0) \end{aligned} \tag{3}$$

or in the matrix form,

$$\underline{x}(t) = \left[\underline{\underline{I}} + \dot{\gamma}(t)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\underline{x}(0)$$

This is the prototype of a weak flow. If we label $\underline{x}(0) = \underline{\xi}$, then at any time t^*, we have recovered the pure shear deformation,

$$\underline{x}(t^*) = \left[\underline{\underline{I}} + \dot{\gamma}t^*\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\underline{\xi}$$

so the relation is $\gamma = \dot{\gamma}t^*$ when $\dot{\gamma}$ is constant. Therefore we have

$$\underline{x}(t^*) = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\underline{\xi}$$

Moreover, $\frac{\partial \underline{v}}{\partial \underline{x}} = \underline{\underline{\Omega}} + \underline{\underline{D}}$ where

$$\underline{\underline{\Omega}} = \frac{\dot{\gamma}}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \underline{\underline{D}} = \frac{\dot{\gamma}}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4}$$

HOMEWORK

1. Posit Navier-Stokes relation for incompressible viscous fluid

$$\underline{\underline{\tau}} = -\gamma \underline{\underline{I}} + 2\eta \underline{\underline{D}}$$

   Evaluate $\underline{\underline{\tau}}$ in the uniaxial incompressible extensional flow with rate $\dot{\epsilon}$ and the incompressible simple shear flow with rate $\dot{\gamma}$. Define

$$\eta_{ext} = \frac{\tau_{11} - \tau_{22}}{2}, \eta_{shear} = \frac{\tau_{12}}{\dot{\gamma}}$$

   Show $\eta_{ext} = 3\eta_{shear}$.

2. So far in this class we recovered canonical homogenous deformation from their flow analysis. Conversely, we can start from $\underline{\underline{F}} = \frac{\partial \underline{x}}{\partial \underline{\xi}}$ and recover flows with initial value $\underline{\xi} = \underline{x}(0)$, and use $\frac{\partial v}{\partial \underline{x}} = \dot{\underline{\underline{F}}} \cdot \underline{\underline{F}}^{-1}$. Recover homogenous uniaxial extensional and simple shear flows from $\underline{\underline{F}}$ of extension and shear.

   For example, for the extensional deformation,

$$\underline{x}(t) = \begin{pmatrix} \alpha_1(t) & 0 & 0 \\ 0 & \alpha_2(t) & 0 \\ 0 & 0 & \alpha_3(t) \end{pmatrix} \underline{x}(0) \equiv \underline{\underline{F}}(t)\underline{x}(0)$$

   with $\Pi\alpha_i = 1$ for incompressible flow and $\alpha_2 = \alpha_3$ for incompressible uniaxial extensional flow. We see that $\dot{\epsilon}_j = \frac{\dot{\alpha}(j)}{\alpha_j}$, or $\frac{\partial v}{\partial x} \mid_{jj} = \frac{\partial v_j}{\partial x_j} = \frac{\dot{\alpha}_j}{\cancel{\partial \alpha_j}}$

   For ~~your~~ future purpose, summarize all kinematic tensors for uniaxial extension, biaxial extension and simple shear "look-up" table.

3. There are many other homogeneous fluids that devices can achieve. Remember that one Neo-Hook~~ies~~ solid law is

$$\underline{\underline{\tau}} = -\gamma \underline{\underline{I}} + G\underline{\underline{B}}$$

   where $\underline{\underline{B}}$ is Finger tens~~ion~~. Impose $\underline{\underline{B}}$ and γ for shear and $\underline{\underline{B}}$ and ϵ for uniaxial extension to derive $\underline{\underline{\tau}}$. Define

$$\begin{aligned} \frac{\tau_{12}}{\gamma} &\equiv \text{Shear modulus } = G \\ \lim_{\epsilon \to 0} \frac{\tau_{11} - \tau_{22}}{\epsilon} &\equiv \text{Young's modulus} = 3G \end{aligned} \tag{5}$$

Deformations and flows continued

Math 769
2-17-09

Recall $\underline{\underline{\mathbf{F}}} = \underline{\underline{\mathbf{R}}}\underline{\underline{U}} = \underline{\underline{\mathbf{V}}}\underline{\underline{\mathbf{R}}}$ $\quad \underline{\underline{\mathbf{F}}} = \dfrac{\partial \underline{x}}{\partial \underline{\xi}}$

R is pure rotation
U is dilational extension/stretch ratios λ_j along principle axes $\underline{A}_j$
U is extension/stretch ratios λ_j along principle axes $\underline{A}_j$

$$\underline{\underline{\mathbf{U}}} = \underline{\underline{\mathbf{R}}}^t \underline{\underline{\mathbf{V}}}\underline{\underline{\mathbf{R}}} \qquad (\underline{\underline{\mathbf{U}}} - \lambda_j \underline{\underline{\mathbf{I}}})\mathbf{A}_j = \underline{0}$$

Definition – Homogeneous deformations are where F, U, V, and R are constant

Recall: $\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{F}}}\underline{\underline{\mathbf{F}}}^t = \underline{\underline{\mathbf{V}}}^2 \qquad \underline{\underline{\mathbf{C}}} = \underline{\underline{\mathbf{F}}}^t\underline{\underline{\mathbf{F}}} = \underline{\underline{\mathbf{U}}}^2$

have identical principal values λ_j^2

Principle Values

$\underline{\underline{\mathbf{E}}}^{NL}$ Lagrangian Strain Tensor $\dfrac{1}{2}(\underline{\underline{\mathbf{C}}} - \underline{\underline{\mathbf{I}}}) = \dfrac{1}{2}(\underline{\underline{\mathbf{U}}}^2 - \underline{\underline{\mathbf{I}}}) \qquad \dfrac{1}{2}(\lambda_j^2 - 1)$

$\underline{\underline{\mathbf{e}}}^{NL}$ Eularian Strain Tensor $\dfrac{1}{2}(\underline{\underline{\mathbf{I}}} - \underline{\underline{\mathbf{B}}}^{-1}) = \dfrac{1}{2}(\underline{\underline{\mathbf{I}}} - (\underline{\underline{\mathbf{V}}}^{-1})^2) \qquad \dfrac{1}{2}(1 - \lambda_j^{-2})$

Define: Principle strains associated with $\underline{\underline{\mathbf{E}}}^{NL}$ or $\underline{\underline{\mathbf{e}}}^{NL}$ are these principle values.

Strain Invariants of any deformation $\xi \to x$ are symmetric polynomials of degree 1, 2, 3 in λ_j^2 ($\underline{\underline{\mathbf{B}}}$ or $\underline{\underline{\mathbf{C}}}$) or λ_j^{-2} ($\underline{\underline{\mathbf{B}}}^{-1}$ or $\underline{\underline{\mathbf{C}}}^{-1}$)

Recall: Invariants of $\underline{\underline{\mathbf{B}}}$ or $\underline{\underline{\mathbf{C}}}$ are

$$\mathbf{I}_1(\underline{\underline{\mathbf{B}}}) = tr(\underline{\underline{\mathbf{B}}}) = 2\lambda_j^2$$

$$\mathbf{I}_2(\underline{\underline{\mathbf{B}}}) = \frac{1}{2}[tr(\underline{\underline{\mathbf{B}}})^2 - tr(\underline{\underline{\mathbf{B}}}^2)] = \sum_{j \neq k} \left(\lambda_j \lambda_k\right)^2$$

$$\mathbf{I}_3(\underline{\underline{\mathbf{B}}}) = \det(\underline{\underline{\mathbf{B}}}) = \prod \lambda_j^2 = \left(\det \underline{\underline{\mathbf{F}}}\right)^2$$

<u>3 Canonical Homogeneous Deformations:</u>

<u>1. Rotation</u> – give rotation about ξ_3 axis by α

$$\underline{\mathbf{x}} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \frac{\partial \underline{x}}{\partial \underline{\xi}} = \underline{\underline{\mathbf{F}}} = \underline{\mathbf{x}}$$

Polar Decomposition $= \underline{\underline{\mathbf{R}}}^{\alpha}\underline{\underline{\mathbf{I}}} = \underline{\underline{\mathbf{I}}}\underline{\underline{\mathbf{R}}}^{\alpha}$

$$\underline{\underline{\mathbf{U}}} = \underline{\underline{\mathbf{V}}} = \underline{\underline{\mathbf{I}}}$$

Pure Rotational – all principle strains are 0 - no strain in a pure rotation

<u>2. Pure Extension</u>

$$\alpha_1 < 1 \quad \underline{\mathbf{x}} \underset{\text{compress}}{\leftarrow} \underline{\xi} \underset{\text{extend}}{\rightarrow} \underline{\mathbf{x}} \quad \alpha_1 > 1$$

$$\underline{\mathbf{x}} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \underline{\xi} \qquad \begin{aligned} \mathbf{x}_1 &= \alpha_1 \xi_1 \\ \mathbf{x}_2 &= \alpha_2 \xi_2 \\ \mathbf{x}_3 &= \alpha_3 \xi_3 \end{aligned}$$

Volume Preserving $= \det \underline{\underline{\mathbf{F}}} = 1 \Rightarrow \prod \alpha_i = 1$

$$\underline{\underline{\mathbf{F}}} = \frac{\partial \underline{x}}{\partial \underline{\xi}} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

Define:

a. Volume Preserving (incompressible): $\det \underline{\underline{\mathbf{F}}} = 1$, $\alpha_1 \alpha_2 \alpha_3 = 1$

b. Uniaxial Extension: $\alpha_1 > \alpha_2 = \alpha_3$

For uniaxial volume preserving extensional deformations:

$$\alpha_1\alpha_2^2 = 1 \qquad \alpha_2 = \alpha_3 = \frac{1}{\sqrt{\alpha_1}} \qquad \underline{\underline{\mathbf{F}}}_{\text{uni, preserving}} = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{\alpha_1}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{\alpha_1}} \end{pmatrix}$$

extension $\alpha_1 > 1$, compression $\alpha_1 < 1$

c. Biaxial Extension $\alpha_1 \neq \alpha_2 \neq \alpha_3$

$$\underline{\underline{\mathbf{B}}} = \underline{\underline{\mathbf{F}}}^2 = \underline{\underline{\mathbf{C}}} = \begin{pmatrix} \alpha_1^2 & 0 & 0 \\ 0 & \frac{1}{\alpha_1} & 0 \\ 0 & 0 & \frac{1}{\alpha_1} \end{pmatrix}$$

Lagrangian $\frac{1}{2}(\underline{\underline{\mathbf{C}}} - \underline{\underline{\mathbf{I}}}) = \frac{1}{2}\begin{pmatrix} \alpha_1^2 - 1 & 0 & 0 \\ 0 & 1 - \frac{1}{\alpha_1} & 0 \\ 0 & 0 & 1 - \frac{1}{\alpha_1} \end{pmatrix}$

Principal Strains are diagonal entries

Eulerian $\frac{1}{2}(\underline{\underline{\mathbf{I}}} - \underline{\underline{\mathbf{B}}}^{-1}) = \frac{1}{2}\begin{pmatrix} 1 - \alpha_1^2 & 0 & 0 \\ 0 & 1 - \alpha_1 & 0 \\ 0 & 0 & 1 - \alpha_1 \end{pmatrix}$

$$\mathbf{I}_1(\underline{\underline{\mathbf{B}}}) = tr(\underline{\underline{\mathbf{B}}}) = \alpha_1^2 + \frac{2}{\alpha_1}$$

Invariants $\mathbf{I}_2(\underline{\underline{\mathbf{B}}}) = \frac{1}{2}[\mathbf{I}_1^2 - tr(\underline{\underline{\mathbf{B}}}^2)] = 2\alpha_1 + \frac{1}{\alpha_1^2}$

$$\mathbf{I}_3(\underline{\underline{\mathbf{B}}}) = 1$$

3. Pure Shear

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 _________        ξ2  s  _______
|         |     | θ /      /
|         |→ h |  /      /
|_________|     |/______/ξ1
                ξ3  out of page
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$$\xi_3 \rightarrow \mathbf{x}_3$$
$$\xi_2 \rightarrow \mathbf{x}_2 \qquad\qquad \tan(\theta) = \frac{\Delta}{\xi_2} = \frac{\mathrm{s}}{\mathrm{h}} = \gamma$$
$$\xi_1 \rightarrow \mathbf{x}_1 = \xi_1 + \Delta(\xi_2) = \xi_1 + \gamma\xi_2$$

Experimental Control = Shear Strain = γ

$$\Delta = \gamma\xi_2 = \frac{\mathrm{s}}{\mathrm{h}}\xi_2 = \tan(\theta)\xi_2$$

$$\underline{\mathbf{x}} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\underline{\xi} \qquad \frac{\partial \underline{x}}{\partial \underline{\xi}} = \underline{\underline{\mathbf{F}}}^{\text{shear}} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \det \underline{\underline{\mathbf{F}}}^{\text{shear}} = 1$$

$$\underline{\underline{\mathbf{B}}}^{\text{shear}} = \underline{\underline{\mathbf{F}}}\,\underline{\underline{\mathbf{F}}}^{\mathrm{t}} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \underline{\underline{\mathbf{C}}}^{\text{shear}} = \underline{\underline{\mathbf{F}}}^{\mathrm{t}}\underline{\underline{\mathbf{F}}} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Invariants
$$\mathbf{I}_1(\underline{\underline{\mathbf{B}}}) = 3+\gamma^2$$
$$\mathbf{I}_2(\underline{\underline{\mathbf{B}}}) = 3+\gamma^2 = \mathbf{I}_1(\underline{\underline{\mathbf{B}}})$$
$$\mathbf{I}_3(\underline{\underline{\mathbf{B}}}) = 1$$

Lagrangian
$$\frac{1}{2}(\underline{\underline{\mathbf{C}}} - \underline{\underline{\mathbf{I}}}) = \begin{pmatrix} 0 & \gamma & 0 \\ \gamma & \gamma^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Eularian
$$\frac{1}{2}(\underline{\underline{\mathbf{I}}} - \underline{\underline{\mathbf{B}}}^{-1}) = \begin{pmatrix} 0 & \gamma & 0 \\ \gamma & -\gamma^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \underline{\underline{\mathbf{B}}}^{-1} = \begin{pmatrix} 1 & -\gamma & 0 \\ -\gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Facts: Principal Values λ_j^2 of $\underline{\underline{\mathbf{C}}}$ and $\underline{\underline{\mathbf{B}}}$ are

$$\lambda_1^2 = \sec\beta + \tan\beta$$
$$\lambda_2^2 = 1 \qquad \text{where} \qquad \tan\beta = \frac{1}{2}\tan\theta = \frac{1}{2}\gamma$$
$$\lambda_3^2 = \sec\beta - \tan\beta$$

Principal Axes of $\underline{\underline{C}}$ are

$$\lambda_1^2 = \begin{pmatrix} \frac{1}{\sqrt{2}}(1-\sin\beta)^{\frac{1}{2}} \\ \frac{1}{\sqrt{2}}(1+\sin\beta)^{\frac{1}{2}} \\ 0 \end{pmatrix} \qquad \lambda_2^2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad \lambda_3^2 = \begin{pmatrix} \frac{1}{\sqrt{2}}(1+\sin\beta)^{\frac{1}{2}} \\ \frac{1}{\sqrt{2}}(1-\sin\beta)^{\frac{1}{2}} \\ 0 \end{pmatrix}$$

"Neo-Hookean" solids (non-linear) are defined by any constitutive law where

$$\underline{\underline{\sigma}} = \underline{\underline{\sigma}}(\underline{\underline{e}}) \text{ or } \underline{\underline{\sigma}}(\underline{\underline{E}})$$

Eularian Strain Tensor or Lagrangian Strain Tensor

$$\otimes: \ \underline{\underline{\sigma}} = \lambda \operatorname{tr}(\underline{\underline{\mathbf{e}}})\underline{\underline{\mathbf{I}}} + 2\mu\underline{\underline{\mathbf{e}}} \qquad \otimes^{-1}: \ \underline{\underline{\mathbf{e}}} = \frac{1+\upsilon}{\mathbf{E}})\underline{\underline{\sigma}} - \frac{\upsilon}{\mathbf{E}}\operatorname{tr}(\underline{\underline{\sigma}})\underline{\underline{\mathbf{I}}}$$

E = Young's Modulus = $2G(1+\upsilon)$ $\qquad$ υ = Poisson's Ratio

Homework: Use $\otimes$ and evaluate $\underline{\underline{\sigma}}$ for 3 canonical deformations -

Rotation by α about 3 axes. Any Pure extension or pure shear, then specialize to $\alpha_1\alpha_2\alpha_3 = 1$

to summarize how σ scales with parameters of deformation.

α ; $\alpha_1, \alpha_2 = \alpha_3 = \dfrac{1}{\sqrt{\alpha_1}}$; γ

<u>Alternative "Neo-Hookean" laws:</u>

$\underline{\underline{\sigma}} = -p\underline{\underline{\mathbf{I}}} + G\underline{\underline{\mathbf{B}}}$ $\qquad$ p=pressure, G=some modulus $G(I_1, I_2, I_3(\underline{\underline{\mathbf{B}}}))$

MATH769-Lecture

Shengqian Chen

February 24, 2009

In this class, we are going to discuss three canonical homogenous flows. The initial position of a fluid particle is $\underline{\xi}$. After time t, the new location is $\underline{x}(\underline{\xi}, t)$ which satisfies the initial condition $\underline{x}(\underline{\xi}, 0) = \underline{\xi}$.

1 Extensional flow

The velocity field for an arbitrary extensional flow is given by

$$\underline{v} = \begin{pmatrix} \dot{\epsilon}_1(t) & 0 & 0 \\ 0 & \dot{\epsilon}_2(t) & 0 \\ 0 & 0 & \dot{\epsilon}_3(t) \end{pmatrix} \underline{x} \equiv \underline{\underline{L}}(t)\underline{x}$$

where $\dot{\epsilon}_j(t)$ are extensional rates. This is equivalent to

$$\begin{aligned} v_{x_1} &= v_1 = \dot{\epsilon}_1(t)x_1 \\ v_{x_2} &= v_2 = \dot{\epsilon}_2(t)x_2 \\ v_{x_3} &= v_3 = \dot{\epsilon}_3(t)x_3 \end{aligned} \tag{1}$$

Since $\underline{v} = \dot{\underline{x}}$, this is actually an ODE system written as $\dot{x}_j = \dot{\epsilon}(t)x_j$. When $\dot{\epsilon}_j(t)$ are constant, the position of the particle can be solved by integration $x_j(t) = x_j(0)e^{\dot{\epsilon}t}$. In general, however, $\dot{\epsilon}_j$ are time dependent then

$$x_j(t) = x_j(0)e^{\int_0^t \dot{\epsilon}_j(s)ds}$$

Remarks

1. At any time t^*, we have

$$\underline{x}(t^*) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix} \underline{x}(0)$$

where $\alpha_j = e^{\dot{\epsilon}_j(t^*)}$ when $\dot{\epsilon}_j$ are constant, so for volume preserving (incompressible) flow,

$$div(\underline{v}) = tr(\underline{\underline{L}}) \equiv \sum \dot{\epsilon}_j = 0$$

and

$$\prod \alpha_j = e^{\sum \dot{\epsilon}_j t^*} = 1$$

2. For the incompressible, uniaxial homogeneous flow with extension along "1" axis, the fact that
$$\dot{\epsilon}_1 > 0, \dot{\epsilon}_2 = \dot{\epsilon}_3 = -\frac{1}{2}\dot{\epsilon}_1$$
recovers the previous relations for uniaxial volume-preserving deformations:
$$\alpha_1 > 1, \alpha_2 = \alpha_3 = \frac{1}{\sqrt{\alpha_1}}$$

3. When the extensional flow has constant rates, i.e. $\dot{\epsilon}_j$ = constant, the extensional flow is a prototype of strong flow because the particle trajectory is exponential:
$$\frac{\partial x_j(t)}{\partial x_j(0)} = e^{\dot{\epsilon}_j(t)}$$

Note It is easy to find the special velocity gradient tensors for the uniaxial, incompressible, extensional flow with constant rate $\dot{\epsilon}(t)$. Since

$$\underline{v} = \dot{\epsilon}_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \underline{x}$$

we have

$$\frac{\partial \underline{v}}{\partial \underline{x}} = \dot{\epsilon}_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix}$$

therefore $\underline{\underline{D}} = \frac{\partial \underline{v}}{\partial \underline{x}}$ and $\underline{\underline{\Omega}} = \underline{\underline{0}}$, indicating the flow has no rotation.

2 Rotational flow

The second canonical homogenous flow is rotation, which we illustrate for rotation about the x_3 axis.

$$\underline{v} = \underline{\dot{x}} = \begin{pmatrix} 0 & \omega(t) & 0 \\ -\omega(t) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \underline{x} \equiv \underline{\underline{\Omega}}(t)\underline{x}$$

where $\omega(t)$ is constant or time-dependent, then $\underline{\underline{D}} = \underline{\underline{0}}$.

HOMEWORK Derive the particle trajectory formula for $\omega(t)$ constant or time-dependent function. Then recover the rotational deformation

$$\underline{x}(t) = \underline{\underline{\Omega}}(t)\underline{x}(0)$$

and relate $\underline{\underline{\Omega}}$ with $\underline{\underline{F}}$ in deformation.

3 Simple Shear

The third canonical homogenous flow is simple(or linear) shear,

$$\underline{v} = \dot{\gamma}(t)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\underline{x}$$

Substitute $\underline{\dot{x}} = \underline{v}$ and solve the ODE system then get the particle trajectory equations when $\dot{\gamma}(t)$ is constant,

$$\begin{aligned} x_3 &= x_3(0) \\ x_2 &= x_2(0) \\ x_1 &= \dot{\gamma}x_2(0)t + x_1(0) \end{aligned} \tag{2}$$

or in the matrix form,

$$\underline{x}(t) = \left[\underline{\underline{I}} + \dot{\gamma}(t)\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\underline{x}(0)$$

This is the prototype of a weak flow. If we label $\underline{x}(0) = \underline{\xi}$, then at any time t^*, we have recovered the pure shear deformation,

$$\underline{x}(t^*) = \left[\underline{\underline{I}} + \dot{\gamma}t^*\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right]\underline{\xi}$$

so the relation is $\gamma = \dot{\gamma}t^*$ when $\dot{\gamma}$ is constant. Therefore we have

$$\underline{x}(t^*) = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\underline{\xi}$$

Moreover, $\frac{\partial \underline{v}}{\partial \underline{x}} = \underline{\underline{\Omega}} + \underline{\underline{D}}$ where

$$\underline{\underline{\Omega}} = \frac{\dot{\gamma}}{2}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \underline{\underline{D}} = \frac{\dot{\gamma}}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

HOMEWORK

1. Posit Navier-Stokes relation for incompressible viscous fluid

$$\underline{\underline{\tau}} = -\gamma\underline{\underline{I}} + 2\eta\underline{\underline{D}}$$

Evaluate $\underline{\underline{\tau}}$ in the uniaxial incompressible extensional flow with rate $\dot{\epsilon}$ and the incompressible simple shear flow with rate $\dot{\gamma}$. Define

$$\eta_{ext} = \frac{\tau_{11} - \tau_{22}}{2}, \eta_{shear} = \frac{\tau_{12}}{\dot{\gamma}}$$

Show $\eta_{ext} = 3\eta_{shear}$.

2. So far in this class we recovered canonical homogenous deformation from their flow analysis. Conversely, we can start from $\underline{\underline{F}} = \frac{\partial \underline{x}}{\partial \underline{\xi}}$ and recover flows with initial value $\underline{\xi} = \underline{x}(0)$, and use $\frac{\partial \underline{v}}{\partial \underline{x}} = \dot{\underline{\underline{F}}} \cdot \underline{\underline{F}}^{-1}$. Recover homogenous uniaxial extensional and simple shear flows from $\underline{\underline{F}}$ of extension and shear.

   For example, for the extensional deformation,

$$\underline{x}(t) = \begin{pmatrix} \alpha_1(t) & 0 & 0 \\ 0 & \alpha_2(t) & 0 \\ 0 & 0 & \alpha_3(t) \end{pmatrix} \underline{x}(0) \equiv \underline{\underline{F}}(t)\underline{x}(0)$$

   with $\Pi\alpha_i = 1$ for incompressible flow and $\alpha_2 = \alpha_3$ for incompressible uniaxial extensional flow. We see that $\dot{\epsilon}_j = \frac{\dot{\alpha}(j)}{\alpha_j}$, or $\frac{\partial v}{\partial x} |_{jj} = \frac{\partial v_j}{\partial x_j} = \frac{\dot{\alpha}_j}{\alpha_j}$

   For future purposes, summarize all kinematic tensors for uniaxial extension, biaxial extension and simple shear so that you have a "look-up" table.

3. There are many other homogeneous fluids that devices can achieve. Remember that one Neo-Hookean solid law is

$$\underline{\underline{\tau}} = -\gamma \underline{\underline{I}} + G\underline{\underline{B}}$$

   where $\underline{\underline{B}}$ is Finger tensor. Impose $\underline{\underline{B}}$ and γ for shear and $\underline{\underline{B}}$ and ϵ for uniaxial extension to derive $\underline{\underline{\tau}}$. Define

$$\begin{aligned} \frac{\tau_{12}}{\gamma} &\equiv \text{Shear modulus } = G \qquad (3) \\ \lim_{\epsilon \to 0} \frac{\tau_{11} - \tau_{22}}{\epsilon} &\equiv \text{Young's modulus} = 3G \end{aligned}$$